GAFA, Geom. funct. anal.
Vol. 15 (2005) 534 – 554
1016-443X/05/020534-21
DOI 10.1007/s00039-005-0515-x

GAFA Geometric And Functional Analysis

ISOPERIMETRIC INEQUALITIES OF EUCLIDEAN TYPE IN METRIC SPACES

S. Wenger

1 Introduction

The purpose of this paper is to prove an isoperimetric inequality of Euclidean type for complete metric spaces admitting a cone-type inequality. These include all Banach spaces and all complete, simply-connected metric spaces of non-positive curvature in the sense of Alexandrov or Busemann. as a consequence we obtain solutions to the Plateau problem in certain spaces, such as duals of Banach spaces and complete simply-connected metric spaces of non-positive curvature. The main theorem generalizes results of Gromov [G] and Ambrosio–Kirchheim [AmK].

1.1 Statement of the main results. The isoperimetric problem of Euclidean type for a space X and given classes $\mathbf{I}_{k-1}$, $\mathbf{I}_k$, and $\mathbf{I}_{k+1}$ of surfaces of dimension $k-1$, k, and $k+1$ in X, together with boundary operators $\mathbf{I}_{k+1} \xrightarrow{\partial} \mathbf{I}_k \xrightarrow{\partial} \mathbf{I}_{k-1}$ and a volume function $\mathbf{M}$ on each class, asks the following: Does there exist for every surface $T \in \mathbf{I}_k$ without boundary, $\partial T = 0$, a surface $S \in \mathbf{I}_{k+1}$ with $\partial S = T$ and such that

$$\mathbf{M}(S) \leq D\mathbf{M}(T)^{(k+1)/k} \tag{1}$$

for a constant D depending only on X and k? A space for which this holds is said to admit an isoperimetric inequality of Euclidean type for $\mathbf{I}_k$ (or in dimension k). The isoperimetric problem of Euclidean type was resolved by Federer and Fleming in [FF] for Euclidean space $X = \mathbb{R}^n$ and in the class $\mathbf{I}_k$ of k-dimensional integral currents, $k \in \{1, \dots, n\}$. In [G] Gromov extended the result to finite dimensional normed spaces and moreover to complete Riemannian manifolds admitting a cone-type inequality (for which the definition will be given below). Gromov worked in the class of Lipschitz chains, formal finite sums of Lipschitz maps on standard simplices. Recently, Ambrosio and Kirchheim extended in [AmK] the theory of currents from the Euclidean setting to general metric spaces. The metric integral currents define suitable classes $\mathbf{I}_k(X)$ of k-dimensional surfaces in X. (It is to be noted that there are metric spaces for which $\mathbf{I}_k(X)$ only consists of the trivial current. However, for the spaces considered here, this is not the case (see below)). In [AmK] the isoperimetric inequality of Euclidean type is proved for dual Banach spaces X admitting an approximation by finite dimensional subspaces in the following sense: There exists a sequence of projections $P_n : X \to X_n$ onto finite dimensional subspaces such that $P_n(x)$ weakly*-converges to x for every $x \in X$. The authors then raise the question whether all Banach spaces admit an isoperimetric inequality of Euclidean type.

In this paper we answer this question affirmatively and, in fact, prove the Euclidean isoperimetric inequality for a large class of metric spaces including also many non-linear ones. We will work in the class of metric integral currents $\mathbf{I}_k(X)$ developed in [AmK], the main definitions of which will be recalled in section 2.2. The intuitive picture of a k-dimensional integral current T one might have (for the moment) is that of a generalized k-dimensional surface with a multiplicity function and an orientation of suitably defined tangent spaces (these will be k-dimensional normed spaces). The orientation defines the boundary ∂T of T which will be a $(k-1)$-dimensional integral current. The volume of T, denoted $\mathbf{M}(T)$ and called mass of T, is the L^1-norm of the multiplicity function with respect to a suitably defined 'Finsler' volume. (T will be a functional rather than a set. The set on which T will live is called the support, denoted $\operatorname{spt} T$). See section 2.2 and [AmK] for the precise definitions. An integral current T with $\partial T = 0$ will be called a cycle.

Definition 1.1. *A metric space (X, d) is said to admit a k-dimensional cone-type inequality (or to admit a cone-type inequality for $\mathbf{I}_k(X)$) if for every cycle $T \in \mathbf{I}_k(X)$ with bounded support there exists an $S \in \mathbf{I}_{k+1}(X)$ satisfying $\partial S = T$ and*

$$\mathbf{M}(S) \leq C_k \operatorname{diam}(\operatorname{spt} T)\mathbf{M}(T)$$

for a constant C_k depending only on k and X.

The main result can be stated as follows:

Theorem 1.2. *Let (X, d) be a complete metric space and $k \in \mathbb{N}$. Suppose that X satisfies a cone-type inequality for $\mathbf{I}_k(X)$ and, if $k \geq 2$, that X also satisfies an isoperimetric inequality of Euclidean type for $\mathbf{I}_{k-1}(X)$. Then (X, d) admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(X)$: For every cycle $T \in \mathbf{I}_k(X)$ there exists an $S \in \mathbf{I}_{k+1}(X)$ with $\partial S = T$ and such that*

$$\mathbf{M}(S) \leq D_k[\mathbf{M}(T)]^{(k+1)/k}$$

where D_k only depends on k and the constants of the cone-type inequality in $\mathbf{I}_k(X)$ and the isoperimetric inequality in $\mathbf{I}_{k-1}(X)$.

We would like to point out that no additional assumptions are made on T. In particular, T can have unbounded support. However, if T has bounded (compact) support then there exists an S satisfying the properties above and which also has bounded (compact) support. The theorem extends [G, 3.4.C] from the setting of Riemannian manifolds to that of complete metric spaces. As will be shown in section 2.3 Banach spaces admit cone-type inequalities in every dimension. This leads to the following generalization of the result of Ambrosio and Kirchheim and answers the question raised.

COROLLARY 1.3. *There are universal constants D_k, $k \geq 1$, such that every Banach space E admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(E)$ with constant D_k.*

In [G, Theorem 4.2.A] Gromov proves an isoperimetric inequality for Lipschitz cycles in arbitrary Banach spaces. The proof uses the fact that Lipschitz cycles admit an approximation by Lipschitz cycles with supports in finite dimensional subspaces. It is not clear whether arbitrary integral currents can be approximated in a similar way.

The constants D_k in Theorem 1.2 and Corollary 1.3 can be computed explicitly. However, they are not optimal, not even if one takes $X = \mathbb{R}^n$. Proving optimality of constants has turned out to be a challenging task. In a major advance Almgren Jr. proved the isoperimetric inequality with optimal constants for $\mathbb{R}^n$ in [A], yielding equality in (1) precisely for round k-dimensional spheres in $(k+1)$-dimensional affine subspaces. For domains in Hadamard manifolds of dimension 3 and 4 the isoperimetric inequality with optimal Euclidean constants has been proved by Kleiner [Kl1] and Croke [C], respectively.

The cone-type inequality is for example satisfied in spaces (X, d) admitting a γ-convex bicombing, for some $\gamma > 0$. By this we mean choices, for all $x, y \in X$, of $\gamma d(x, y)$-Lipschitz paths $c_{xy} : [0, 1] \to X$ joining x to y and such that the following holds: For any three points $u, v, v' \in X$ the inequality

$$d\big(c_{uv}(t), c_{uv'}(t)\big) \leq \gamma d(v, v')$$

holds for all $t \in [0, 1]$. Examples of such spaces include all complete simply-connected metric spaces of non-positive curvature in the sense of Alexandrov (called Hadamard spaces in the sequel) and, more generally, spaces with a convex metric. The definitions will be given in section 2.1. It should be mentioned here that these spaces contain many rectifiable sets (see [Kl2]) and hence $\mathbf{I}_k(X)$ is not trivial.

Corollary 1.4. *For fixed $\gamma > 0$ and $k \in \mathbb{N}$, every complete metric space (X, d) with a γ-convex bicombing admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(X)$ with constants D_k depending only on k and γ.*

As an application of the above results we will prove the existence of a solution to the generalized Plateau problem in dual Banach spaces and in Hadamard spaces.

Theorem 1.5. *If E is the dual of a Banach space then for every $T \in \mathbf{I}_k(E)$ with compact support and $\partial T = 0$ there exists an $S \in \mathbf{I}_{k+1}(E)$ with $\partial S = T$ and*

$$\mathbf{M}(S) = \inf\left\{\mathbf{M}(S') : S' \in \mathbf{I}_{k+1}(E), \partial S' = T\right\}. \tag{2}$$

Moreover, every $S \in \mathbf{I}_{k+1}(E)$ which satisfies $\partial S = T$ and (2) has compact support.

This extends [AmK, Theorem 10.6]. We point out that we do not make the assumption that the predual of E be separable. In [AmK] there are examples of non-dual spaces for which the Plateau problem has a solution. For general Banach spaces, the Plateau problem is unsolved.

Concerning (non-linear) metric spaces we have the following result.

Theorem 1.6. *If (X, d) is a Hadamard space then for every $T \in \mathbf{I}_k(X)$ with compact support and $\partial T = 0$ there exists an $S \in \mathbf{I}_{k+1}(X)$ with $\partial S = T$ and*

$$\mathbf{M}(S) = \inf\left\{\mathbf{M}(S') : S' \in \mathbf{I}_{k+1}(X), \partial S' = T\right\}. \tag{3}$$

Moreover, every $S \in \mathbf{I}_{k+1}(X)$ which satisfies $\partial S = T$ and (3) has compact support.

The two theorems above will follow from a more general theorem which uses the ultra-completion of metric spaces (section 4). The basic ideas in the proof are similar to those in [AmK, Theorem 10.6]. The argument using ultra-completions is due to Urs Lang.

1.2 Outline of the main argument. The proof of Theorem 1.2 is inspired by Gromov's argument. However, the methods in [G] rely in several ways on the bi-Lipschitz embeddability of compact Riemannian manifolds into Euclidean space. Our approach uses a more intrinsic analysis of k-dimensional cycles. For the description of our argument it is convenient to introduce the following terminology: A cycle $T \in \mathbf{I}_k(X)$ is called *round* if

$$\operatorname{diam}(\operatorname{spt} T) \leq E\mathbf{M}(T)^{1/k}$$

for a constant E depending only on k and on the space X. The essential step in the proof is stated in Proposition 3.1 which claims the following: Under the hypotheses of Theorem 1.2 every cycle $T \in \mathbf{I}_k(X)$ can be decomposed into the sum $T = T_1 + \cdots + T_N + R$ of round cycles T_i and a cycle R with the properties that $\sum \mathbf{M}(T_i) \leq (1+\lambda)\mathbf{M}(T)$ and $\mathbf{M}(R) \leq (1-\delta)\mathbf{M}(T)$ for constants $0 < \delta, \lambda < 1$ depending only on k and the constant from the isoperimetric inequality for $\mathbf{I}_{k-1}(X)$. The construction of such a decomposition is based on an analysis of the growth of the function $r \mapsto \|T\|(B(y,r))$. Intuitively speaking, $\|T\|(B(y,r))$ is the volume of T lying in the closed ball $B(y,r)$ with center y and radius r (see section 2.2 for the precise definition of $\|T\|$). For almost every $y \in \operatorname{spt} T$ we have

$$\|T\|\big(B(y,r)\big) \geq Fr^k \tag{4}$$

for small $r > 0$ and a constant F depending only on k. Denoting by $r_0(y)$ the least upper bound of those r satisfying (4) one can prove that, when T is cut open along the metric sphere with center y and of radius about $r_0(y)$ only little boundary is created. By closing this boundary with a suitable isoperimetric filling (Lemma 3.4) one constructs a decomposition of T into a sum $T = T_1 + \tilde{R}$. The cycle T_1, lying essentially in $B(y, r_0(y))$ is round. This will easily follow from the definition of $r_0(y)$. By using a simple Vitali-type covering argument one then shows that enough such round cycles T_i can be split off in order to leave a rest R satisfying $\mathbf{M}(R) \leq (1-\delta)\mathbf{M}(T)$. Successive application of Proposition 3.1 will easily establish the proof of the main theorem.

The paper is structured as follows: In section 2.2 we recall the main definitions from the theory of currents in metric spaces and state those results from [AmK] vital for our purposes. Then, following a construction in [AmK], we prove cone-type inequalities for various classes of metric spaces. The decomposition of a given cycle is constructed in section 3. This forms the main part of the paper. Section 3 also contains the proof of the main theorem. The last section contains a general theorem from which Theorem 1.5 and Theorem 1.6 will be derived.

Acknowledgements. I would like to thank Urs Lang for helpful discussions on the subject, and in particular for the contribution to the Plateau problem for Hadamard spaces. My thanks also go to Luigi Ambrosio for comments concerning the final version of the paper.

2 Preliminaries

In section 2.1 we review the definition of a convex bicombing on a metric space and give a list of spaces admitting a convex bicombing. One important class is that of simply-connected metric spaces of non-positive curvature in the sense of Alexandrov for which the definition will also be given. Section 2.2 contains the main definitions from the theory of metric currents developed in [AmK] as well as the results relevant in our context. The purpose of section 2.3 is to construct cone fillings and to prove a cone-type inequality for metric spaces admitting a convex bicombing. The preliminary material on ultra-completions needed for solving the Plateau problem is to be found in section 4.

2.1 Convex bicombings and spaces of non-positive curvature. A continuous curve $c : [0,1] \to X$ in a metric space (X,d) is called rectifiable if it has finite length, i.e. if

$$\mathrm{length}(c) := \sup\left\{ \sum_{i=1}^{N-1} d\big(c(t_i), c(t_{i+1})\big) : 0 = t_1 < \cdots < t_N = 1 \right\} < \infty .$$

An isometric embedding $c : [0, d(x,y)] \to X$ satisfying $c(0) = x$ and $c(d(x,y)) = y$ is called a geodesic segment from x to y. The space X is called geodesic if every two points x and y can be connected by a geodesic segment.

Definition 2.1. *A γ-convex bicombing on a metric space (X,d) is a choice of curves $c_{xy} : [0,1] \to X$ joining x to y, for each two points $x, y \in X$, such that the following two conditions hold:*

(i) *For any points $u, v \in X$ the curve c_{uv} is Lipschitz with Lipschitz constant $\gamma d(u,v)$.*

(ii) *For any three points $u, v, v' \in X$ and for $t \in [0,1]$ we have*

$$d\big(c_{uv}(t), c_{uv'}(t)\big) \le \gamma d(v, v') .$$

If a metric space has a γ-convex bicombing for some $\gamma > 0$ then it is said to admit a convex bicombing. Let $\varphi : X \to Y$ be a bi-Lipschitz homeomorphism between two metric spaces X and Y. Then X admits a convex bicombing if and only if Y does. In this sense, the above definition is bi-Lipschitz invariant.

Example. *For a normed space E the straight lines $c_{xy}(t) := ty + (1-t)x$ where $t \in [0,1]$ and $x, y \in E$ define a 1-convex bicombing.*

Complete simply-connected Alexandrov spaces of non-positive curvature (called Hadamard spaces) form another important class of spaces admitting a convex bicombing. They are defined as follows: Let (X,d) be a metric space. A geodesic triangle in X consists of three points $x, y, z \in X$ and of a choice of three geodesic segments joining them, denoted by $[x,y],[y,z],[z,x]$. A comparison triangle for the geodesic triangle $\Delta([x,y],[y,z],[z,x])$ is a triangle $\overline{\Delta}([\overline{x},\overline{y}],[\overline{y},\overline{z}],[\overline{z},\overline{x}])$ in $\mathbb{R}^2$ with vertices $\overline{x},\overline{y},\overline{z} \in \mathbb{R}^2$ satisfying $d(x,y) = |\overline{x}-\overline{y}|$, $d(y,z) = |\overline{y}-\overline{z}|$, and $d(z,x) = |\overline{z}-\overline{x}|$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a comparison point for $p \in [x,y]$ if $d(x,p) = |\overline{x}-\overline{p}|$. Comparison points for points on the other geodesic sides of the triangle are defined similarly.

DEFINITION 2.2. *A geodesic metric space (X,d) is called* CAT(0)*-space if for every geodesic triangle Δ and its comparison triangle $\overline{\Delta}$, and for all points $p,q \in \Delta$ the comparison points $\overline{p},\overline{q} \in \overline{\Delta}$ satisfy*

$$d(p,q) \leq |\overline{p}-\overline{q}|\,.$$

A complete CAT(0)-space is called Hadamard space. We refer to [BH] for an account on these spaces.

EXAMPLE. *Every complete simply-connected Riemannian manifold of non-positive sectional curvature is a Hadamard space.*

CAT(0)-spaces clearly admit a convex bicombing. Indeed, every two points can be joined by a unique geodesic and for points u,v,v' the reparametrizations c_{uv} and $c_{uv'}$ (to $[0,1]$) of the geodesics joining u to v and, respectively, u to v' satisfy

$$d\big(c_{uv}(t), c_{uv'}(t)\big) \leq t d(v,v') \quad \text{for all } t \in [0,1]\,. \tag{5}$$

A uniquely geodesic metric space X for which (5) holds for all points $u,v,v' \in X$ is said to have a convex metric. Such a space clearly admits a convex bicombing.

2.2 Currents in metric spaces. The general reference for this section is [AmK]. Let (X,d) be a complete metric space and let $\mathcal{D}^k(X)$ denote the set of $(k+1)$-tuples $(f,\pi_1,\dots,\pi_k)$ of Lipschitz functions on X with f bounded. The Lipschitz constant of a Lipschitz function f on X will be denoted by $\mathrm{Lip}(f)$.

DEFINITION 2.3. *A k-dimensional metric current T on X is a multi-linear functional on $\mathcal{D}^k(X)$ satisfying the following properties:*

(i) *If π_i^j converges pointwise to π_i as $j \to \infty$ and if $\sup_{i,j} \mathrm{Lip}(\pi_i^j) < \infty$ then*

$$T(f,\pi_1^j,\dots,\pi_k^j) \longrightarrow T(f,\pi_1,\dots,\pi_k)\,.$$

(ii) *If $\{x \in X : f(x) \neq 0\}$ is contained in the union $\bigcup_{i=1}^k B_i$ of Borel sets B_i and if π_i is constant on B_i then*

$$T(f,\pi_1,\dots,\pi_k) = 0\,.$$

(iii) *There exists a finite Borel measure μ on X such that*

$$\left|T(f,\pi_1,\dots,\pi_k)\right| \leq \prod_{i=1}^{k} \mathrm{Lip}(\pi_i) \int_X |f| d\mu \tag{6}$$

for all $(f,\pi_1,\dots,\pi_k) \in \mathcal{D}^k(X)$.

The space of k-dimensional metric currents on X is denoted by $\mathbf{M}_k(X)$ and the minimal Borel measure μ satisfying (6) is called mass of T and written as $\|T\|$. We also call mass of T the number $\|T\|(X)$ which we denote by $\mathbf{M}(T)$. The support of T is, by definition, the closed set $\operatorname{spt} T$ of points $x \in X$ such that $\|T\|(B(x,r)) > 0$ for all $r > 0$. Here, $B(x,r)$ denotes the closed ball $B(x,r) := \{y \in X : d(y,x) \leq r\}$.

Remark 2.4. *As is done in [AmK] we will also assume here that the cardinality of any set is an Ulam number. This is consistent with the standard ZFC set theory. We then have that $\operatorname{spt} T$ is separable and furthermore that $\|T\|$ is concentrated on a σ-compact set, i.e. $\|T\|(X\backslash C) = 0$ for a σ-compact set $C \subset X$ (see [AmK]).*

The restriction of $T \in \mathbf{M}_k(X)$ to a Borel set $A \subset X$ is given by

$$(T \llcorner A)(f,\pi_1,\dots,\pi_k) := T(f\chi_A,\pi_1,\dots,\pi_k)\,.$$

This expression is well defined since T can be extended to a functional on tuples for which the first argument lies in $L^\infty(X,\|T\|)$.

The boundary of $T \in \mathbf{M}_k(X)$ is the functional

$$\partial T(f,\pi_1,\dots,\pi_{k-1}) := T(1,f,\pi_1,\dots,\pi_{k-1})\,.$$

It is clear that ∂T satisfies conditions (i) and (ii) in the above definition. If ∂T also has finite mass (condition (iii)) then T is called a normal current. The respective space is denoted by $\mathbf{N}_k(X)$.

The push-forward of $T \in \mathbf{M}_k(X)$ under a Lipschitz map φ from X to another complete metric space Y is given by

$$\varphi_\# T(g,\tau_1,\dots,\tau_k) := T(g \circ \varphi, \tau_1 \circ \varphi, \dots, \tau_k \circ \varphi)$$

for $(g,\tau_1,\dots,\tau_k) \in \mathcal{D}^k(Y)$. This defines a k-dimensional current on Y, as is easily verified.

In this paper we will mainly be concerned with integer rectifiable and integral currents. For notational purposes we first repeat some well-known definitions. The Hausdorff k-dimensional measure of $A \subset X$ is defined to be

$$\mathcal{H}^k(A) := \lim_{\delta\searrow 0} \inf\left\{ \sum_{i=1}^{\infty} \omega_k \left(\frac{\mathrm{diam}(B_i)}{2}\right)^k : B \subset \bigcup_{i=1}^{\infty} B_i,\ \mathrm{diam}(B_i) < \delta \right\},$$

where ω_k denotes the Lebesgue measure of the unit ball in $\mathbb{R}^k$. The k-dimensional lower density $\Theta_{*k}(\mu, x)$ of a finite Borel measure μ at a point x is given by the formula

$$\Theta_{*k}(\mu, x) := \liminf_{r \searrow 0} \frac{\mu(B(x,r))}{\omega_k r^k}\,.$$

An $\mathcal{H}^k$-measurable set $A \subset X$ is said to be countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz maps $f_i : B_i \to X$ from subsets $B_i \subset \mathbb{R}^k$ such that

$$\mathcal{H}^k\Big(A \backslash \bigcup f_i(B_i)\Big) = 0\,.$$

DEFINITION 2.5. *A current $T \in \mathbf{M}_k(X)$ with $k \geq 1$ is said to be rectifiable if*

(i) $\|T\|$ *is concentrated on a countably $\mathcal{H}^k$-rectifiable set and*
(ii) $\|T\|$ *vanishes on $\mathcal{H}^k$-negligible sets.*

T is called integer rectifiable if, in addition, the following property holds:

(iii) *For any Lipschitz map $\varphi\colon X \to \mathbb{R}^k$ and any open set $U \subset X$ there exists $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$ such that*

$$\varphi_{\#}(T \llcorner U)(f, \pi_1, \dots, \pi_k) = \int_{\mathbb{R}^k} \theta f \det\left(\frac{\partial \pi_i}{\partial x_j}\right) d\mathcal{L}^k$$

for all $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(\mathbb{R}^k)$.

A 0-dimensional (integer) rectifiable current is a $T \in \mathbf{M}_0(X)$ of the form

$$T(f) = \sum_{i=1}^{\infty} \theta_i f(x_i)\,, \quad f \text{ Lipschitz and bounded}\,,$$

for suitable $\theta_i \in \mathbb{R}$ (or $\theta_i \in \mathbb{Z}$) and $x_i \in X$.

The space of rectifiable currents is denoted by $\mathcal{R}_k(X)$, that of integer rectifiable currents by $\mathcal{I}_k(X)$. Endowed with the mass norm $\mathbf{M}_k(X)$ is a Banach space, $\mathcal{R}_k(X)$ a closed subspace, and $\mathcal{I}_k(X)$ a closed additive subgroup. This follows directly from the definitions. Integer rectifiable normal currents are called integral currents. The respective space is denoted by $\mathbf{I}_k(X)$. As the mass of a k-dimensional normal current vanishes on $\mathcal{H}^k$-negligible sets [AmK, Theorem 3.9] it is easily verified that the push-forward of an integral current under a lipschitz map is again an integral current. In the following, an element $T \in \mathbf{I}_k(X)$ with zero boundary $\partial T = 0$ will be called a cycle. An element $S \in \mathbf{I}_{k+1}(X)$ satisfying $\partial S = T$ is said to be a filling of T.

The characteristic set S_T of a rectifiable current $T \in \mathcal{R}_k(X)$ is defined by

$$S_T := \left\{x \in X : \Theta_{\star k}(\|T\|, x) > 0\right\}. \tag{7}$$

It can be shown that S_T is countably $\mathcal{H}^k$-rectifiable and that $\|T\|$ is concentrated on S_T. In the next theorem the function $\lambda : S_T \to (0, \infty)$ denotes the area factor on the (weak) tangent spaces to S_T as defined in [AmK]. We do not provide a definition here since for our purposes it is enough to know that λ is $\mathcal{H}^k$-integrable and bounded from below by $k^{-k/2}$ (see [AmK, Lemma 9.2]).

Theorem 2.6 [AmK, Theorem 9.5]**.** *If $T \in \mathcal{R}_k(X)$ then there exists a $\mathcal{H}^k$-integrable function $\theta : S_T \to (0, \infty)$ such that*

$$\|T\|(A) = \int_{A \cap S_T} \lambda\theta d\mathcal{H}^k \qquad \text{for } A \subset X \text{ Borel,}$$

that is, $\|T\| = \lambda\theta d\mathcal{H}^k \llcorner S_T$. Moreover, if T is an integral current then θ takes values in $\mathbb{N} := \{1, 2, \dots\}$ only.

The following slicing theorem (proved in [AmK, Theorems 5.6 and 5.7]) is, besides Theorem 2.6, the only result from the theory of metric currents needed in the proof of the main result.

Theorem 2.7. *Let be $T \in \mathbf{N}_k(X)$ and ϱ a Lipschitz function on X. Then there exists for almost every $r \in \mathbb{R}$ a normal current $\langle T, \varrho, r\rangle \in \mathbf{N}_{k-1}(X)$ with the following properties:*

(i) $\langle T, \varrho, r\rangle = \partial(T \llcorner \{\varrho \leq r\}) - (\partial T) \llcorner \{\varrho \leq r\}$;

(ii) $\|\langle T, \varrho, r\rangle\|$ *and* $\|\partial\langle T, \varrho, r\rangle\|$ *are concentrated on* $\varrho^{-1}(\{r\})$;

(iii) $\mathbf{M}(\langle T, \varrho, r\rangle) \leq \operatorname{Lip}(\varrho)\frac{d}{dr}\mathbf{M}(T \llcorner \{\varrho \leq r\})$.

Moreover, if $T \in \mathbf{I}_k(X)$ then $\langle T, \varrho, r\rangle \in \mathbf{I}_{k-1}(X)$ for almost all $r \in \mathbb{R}$.

2.3 Cone constructions and cone-type inequalities.

The following cone construction is a slightly modified version of the one given in [AmK].

Let (X, d) be a complete metric space and $T \in \mathbf{N}_k(X)$ and endow $[0, 1] \times X$ with the Euclidean product metric. Given a Lipschitz function f on $[0, 1] \times X$ and $t \in [0, 1]$ we define the function $f_t : X \to \mathbb{R}$ by $f_t(x) := f(t, x)$. To every $T \in \mathbf{N}_k(X)$ and every $t \in [0, 1]$ we associate the normal k-current on $[0, 1] \times X$ given by the formula

$$\big([t] \times T\big)(f, \pi_1, \dots, \pi_k) := T(f_t, \pi_{1t}, \dots, \pi_{kt}),$$

The product of a normal current with the interval $[0, 1]$ is defined as follows.

Definition 2.8. *For a normal current $T \in \mathbf{N}_k(X)$ the functional $[0, 1] \times T$ on $\mathcal{D}^{k+1}([0, 1] \times X)$ is given by*

$$\big([0,1]\times T\big)(f,\pi_1,\dots,\pi_{k+1}) \\ := \sum_{i=1}^{k+1}(-1)^{i+1}\int_0^1 T\left(f_t\frac{\partial\pi_{it}}{\partial t},\pi_{1t},\dots,\pi_{i-1t},\pi_{i+1t},\dots,\pi_{k+1t}\right)dt$$

for $(f,\pi_1,\dots,\pi_{k+1})\in\mathcal{D}^{k+1}([0,1]\times X)$.

We now have the following result whose proof is analogous to that of [AmK, Proposition 10.2 and Theorem 10.4].

Theorem 2.9. *For every $T\in\mathbf{N}_k(X)$ with bounded support the functional $[0,1]\times T$ is a $(k+1)$-dimensional normal current on $[0,1]\times X$ with boundary*

$$\partial\big([0,1]\times T\big)=[1]\times T-[0]\times T-[0,1]\times\partial T\,.$$

Moreover, if $T\in\mathbf{I}_k(X)$ then $[0,1]\times T\in\mathbf{I}_{k+1}([0,1]\times X)$.

Proposition 2.10. *If (X,d) is a complete metric space admitting a γ-convex bicombing then every cycle $T\in\mathbf{I}_k(X)$, $k\geq 1$, with bounded support has a filling $S\in\mathbf{I}_{k+1}(X)$ satisfying*

$$\mathbf{M}(S)\leq(k+1)\gamma^{k+1}\operatorname{diam}(\operatorname{spt}T)\mathbf{M}(T)\,.$$

Proof. We fix $x_0\in\operatorname{spt}T$ and define a locally Lipschitz map $\varphi:[0,1]\times X\to X$ by $\varphi(t,x):=c_{x_0x}(t)$. Then, for fixed $x\in\operatorname{spt}T$, the map $t\mapsto\varphi(t,x)$ is $\gamma\operatorname{diam}(\operatorname{spt}T)$-Lipschitz, whereas for fixed $t\in[0,1]$ the map $x\mapsto\varphi(t,x)$ is γ-Lipschitz. Theorem 2.9 implies that $\varphi_\#([0,1]\times T)\in\mathbf{I}_{k+1}(X)$ and furthermore

$$\partial\varphi_\#\big([0,1]\times T\big)=\varphi_\#\big(\partial([0,1]\times T)\big)=\varphi_\#\big([1]\times T\big)-\varphi_\#\big([0]\times T\big)=T\,.$$

To obtain the estimate on mass we compute for $(f,\pi_1,\dots,\pi_{k+1})\in\mathcal{D}^{k+1}(X)$ that

$$\begin{aligned}
&\big|\varphi_\#([0,1]\times T)(f,\pi_1,\dots,\pi_{k+1})\big|\\
&\leq\sum_{i=1}^{k+1}\left|\int_0^1 T\left(f\circ\varphi_t\frac{\partial\pi_i\circ\varphi_t}{\partial t},\pi_1\circ\varphi_t,\dots,\pi_{i-1}\circ\varphi_t,\pi_{i+1}\circ\varphi_t,\dots,\pi_{k+1}\circ\varphi_t\right)dt\right|\\
&\leq\sum_{i=1}^{k+1}\int_0^1\prod_{j\neq i}\operatorname{Lip}(\pi_j\circ\varphi_t)\int_X\left|f\circ\varphi_t\frac{\partial(\pi_i\circ\varphi_t)}{\partial t}\right|d\|T\|dt\\
&\leq(k+1)\gamma^{k+1}\operatorname{diam}(\operatorname{spt}T)\prod_{j=1}^{k+1}\operatorname{Lip}(\pi_j)\int_0^1\int_X\big|f\circ\varphi(t,x)\big|d\|T\|(x)dt\,.
\end{aligned}$$

From this it follows that

$$\varphi_\#\big\|[0,1]\times T\big\|\leq(k+1)\gamma^{k+1}\operatorname{diam}(\operatorname{spt}T)\varphi_\#\big(\mathcal{L}^1\times\|T\|\big)\,,$$

and this concludes the proof. □

3 Partial Decomposition and Proof of the Main Result

The aim of this section is to prove the proposition below which forms the crucial step when decomposing a cycle into the sum of round cycles. This result will be used to prove the Theorem 1.2.

PROPOSITION 3.1. *Let (X, d) be a complete metric space and $k \geq 1$ an integer. If $k \geq 2$ then suppose furthermore that X has a Euclidean isoperimetric inequality for $\mathbf{I}_{k-1}(X)$ with a constant $C > 0$. There then exist constants $E > 0$ and $0 < \delta, \lambda < 1$ depending only on k and C with the following property: Every cycle $T \in \mathbf{I}_k(X)$ admits a decomposition $T = \sum_{i=1}^{N} T_i + R$ into a sum of integral cycles satisfying:*

(i) $\operatorname{diam}(\operatorname{spt} T_i) \leq E\mathbf{M}(T_i)^{1/k}$;
(ii) $\mathbf{M}(R) \leq (1-\delta)\mathbf{M}(T)$;
(iii) $\sum_{i=1}^{N} \mathbf{M}(T_i) \leq (1+\lambda)\mathbf{M}(T)$.

We first state some preparatory lemmas. The first will be employed to obtain the estimate in (ii) for the cycle R.

LEMMA 3.2. *Let (Y, d) be a metric space, μ a finite Borel measure on Y, and $F > 0$, $k \in \mathbb{N}$. For $y \in Y$ define*

$$r_0(y) := \max\left\{r \geq 0 : \mu(B(y, r)) \geq Fr^k\right\}.$$

If $r_0(y) > 0$ for μ-almost every $y \in Y$ then there exist points $y_1, \dots, y_N \in Y$ satisfying

(i) $r_0(y_i) > 0$;
(ii) $B(y_i, 2r_0(y_i)) \cap B(y_j, 2r_0(y_j)) = \emptyset$ *if* $i \neq j$;
(iii) $\sum_{i=1}^{N} \mu(B(y_i, r_0(y_i))) \geq \alpha\mu(Y)$;

for a constant $\alpha > 0$ depending only on k.

The proof, provided for completeness, is analogous to that of the simple version of the Vitali covering lemma.

Proof. Set $Y_1 := Y$ and $r_1^\star := \sup\{r_0(y) : y \in Y_1\}$ and choose $y_1 \in Y_1$ such that $r_0(y_1) > \frac{2}{3}r_1^\star$. If $y_1, \dots, y_j$ are chosen, we define

$$Y_{j+1} := Y \Big\backslash \bigcup_{i=1}^{j} B\big(y_i, 5r_0(y_i)\big)$$

and $r_{j+1}^\star := \sup\{r_0(y) : y \in Y_{j+1}\}$. If $\mu(Y_{j+1}) > 0$ then we choose $y_{j+1} \in Y_{j+1}$ such that $r_0(y_{j+1}) > \frac{2}{3}r_{j+1}^\star$. This procedure yields (possibly finite) sequences $y_i \in Y_i$ and $r_1^\star \geq r_2^\star \geq \dots$ and we claim that $B(y_i, 2r_0(y_i)) \cap B(y_j, 2r_0(y_j)) = \emptyset$ if $i \neq j$. This is immediate since

$$d(y_i, y_j) \geq 5r_0(y_i) > 2r_0(y_i) + 2r_j^\star \geq 2r_0(y_i) + 2r_0(y_j)$$

for $i < j$. If we have $\mu(Y_N) = 0$ for some N then it follows that

$$5^k \sum_{i=1}^{N} \mu\big(B(y_i, r_0(y_i))\big) = \sum_{i=1}^{N} F\big[5r_0(y_i)\big]^k > \sum_{i=1}^{N} \mu\big(B(y_i, 5r_0(y_i))\big) \geq \mu(Y)\,.$$

If, on the other hand, $\mu(Y_N) > 0$ for all $N \in \mathbb{N}$ then it follows that $r_N^\star \to 0$ as $N \to \infty$ and furthermore that $\mu\big(Y \setminus \bigcup_{i=1}^{\infty} B(y_i, 5r_0(y_i))\big) = 0$. The proof then follows as in the finite case. $\square$

The study of the growth of the function $r \mapsto \|T\|(B(x,r))$ will play a predominant role in the proof of Proposition 3.1. In this context the following easy fact will be helpful.

LEMMA 3.3. *Fix $\bar{C} > 0$, $k \geq 2$, $0 \leq r_0 < r_1 < \infty$, and suppose $\beta : [r_0, r_1] \to (0, \infty)$ is non-decreasing and satisfies*

(i) $\beta(r_0) = \frac{r_0^k}{\bar{C}^{k-1}k^k}$;
(ii) $\beta(r) \leq \bar{C}[\beta'(r)]^{k/(k-1)}$ *for a.e.* $r \in (r_0, r_1)$.

Then it follows that

$$\beta(r) \geq \frac{r^k}{\bar{C}^{k-1}k^k} \quad \textit{for all } r \in [r_0, r_1]\,.$$

Proof. By rearranging (ii) we obtain

$$\frac{\beta'(t)}{\beta(t)^{\frac{k-1}{k}}} \geq \frac{1}{\bar{C}^{\frac{k-1}{k}}}$$

and integration from r_0 to r yields the claimed estimate. $\square$

The next statement is concerned with the support of fillings. It will be used to prove the roundness of the cycles T_i. The T_i will be constructed by restricting T to a ball $B(y_i, r)$ and filling in the boundary $\partial(T \llcorner B(y_i, r))$ by a filling satisfying the isoperimetric inequality. Lemma 3.4 ensures that we can choose a filling whose support stays near its boundary.

LEMMA 3.4. *Let (X, d) be a complete metric space and $k \geq 2$. Suppose that X admits a Euclidean isoperimetric inequality for $\mathbf{I}_{k-1}(X)$ with a constant $C > 0$. Then there exists for every cycle $T \in \mathbf{I}_{k-1}(X)$ and every $\varepsilon > 0$ an $S \in \mathbf{I}_k(X)$ satisfying $\partial S = T$ and*

$$\mathbf{M}(S) \leq \inf\big\{\mathbf{M}(S') : S' \in \mathbf{I}_k(X), \partial S' = T\big\} + \varepsilon$$

and furthermore

$$\|S\|\big(B(x,r)\big) \geq \frac{r^k}{(3C)^{k-1}k^k}$$

for all $x \in \operatorname{spt} S$ and $0 \leq r \leq \operatorname{dist}(x, \operatorname{spt} T)$.

In particular, there exists for every cycle $T \in \mathbf{I}_{k-1}(X)$ an $S \in \mathbf{I}_k(X)$ with $\partial S = T$ and

$$\mathbf{M}(S) \le C[\mathbf{M}(T)]^{k/(k-1)}$$

and such that

$$\operatorname{spt} S \subset B\big(\operatorname{spt} T, 3Ck\mathbf{M}(T)^{1/(k-1)}\big)\,.$$

Here, $B(A,\varrho)$ denotes the ϱ-neighborhood of the set A. The proof of the lemma is essentially contained in the proof of [AmK, Theorem 10.6].

Proof. Let $\mathcal{M}$ denote the complete metric space consisting of all fillings $S \in \mathbf{I}_k(X)$ of T and endowed with the metric given by $d_{\mathcal{M}}(S,S') := \mathbf{M}(S-S')$. By the Ekeland variational principle [E], for every $\varepsilon \in (0,1/2)$, there exists an $S \in \mathcal{M}$ satisfying

$$\mathbf{M}(S) \le \inf\big\{\mathbf{M}(S') : S' \in \mathbf{I}_{k+1}(X)\,,\ \partial S' = T\big\} + \varepsilon$$

(and thus the isoperimetric inequality for $\varepsilon > 0$ small enough) and such that the function

$$S' \in \mathcal{M} \mapsto \mathbf{M}(S') + \varepsilon\mathbf{M}(S'-S)$$

is minimal at $S' = S$. Let be $x \in \operatorname{spt} S \setminus \operatorname{spt} T$ and set $\varrho_x(y) := d(x,y)$. Then, for almost every $0 < r < d(x, \operatorname{spt} T)$ the slice $\langle S, \varrho_x, r\rangle$ exists, has zero boundary, and belongs to $\mathbf{I}_{k-1}(X)$. For an isoperimetric filling $S_r \in \mathbf{I}_k(X)$ of $\langle S, \varrho_x, r\rangle$ the integral current $S \llcorner B^c(x,r) + S_r$ has boundary T and thus, comparison with S yields

$$\mathbf{M}\big(S \llcorner B^c(x,r) + S_r\big) + \varepsilon\mathbf{M}\big(S \llcorner B(x,r) - S_r\big) \ge \mathbf{M}(S)\,.$$

Here, $B^c(x,r)$ denotes the complement of the ball $B(x,r)$. Together with the isoperimetric inequality, the above estimate implies that

$$\mathbf{M}\big(S \llcorner B(x,r)\big) \le \frac{1+\varepsilon}{1-\varepsilon}\mathbf{M}(S_r) \le 3C\mathbf{M}\big(\langle S, \varrho_x, r\rangle\big)^{k/(k-1)}$$

for almost every $r \in (0, \operatorname{dist}(x, \operatorname{spt} T))$. Setting $\beta(r) := \|S\|(B(x,r))$ and using the slicing theorem we obtain the inequality

$$\beta(r) \le 3C\big[\beta'(r)\big]^{k/(k-1)} \text{ for a.e. } r \in \big(0, \operatorname{dist}(x, \operatorname{spt} T)\big)$$

which, after applying Lemma 3.3, yields

$$\|S\|\big(B(x,r)\big) \ge \frac{r^k}{(3C)^{k-1}k^k} \quad \text{for all } 0 \le r < \operatorname{dist}(x, \operatorname{spt} T)\,.$$

This proves the lemma. □

We are now ready to prove the proposition.

Proof of Proposition 3.1. First of all, let F be given by $F := \lambda^{k-1}/C^{k-1}k^k$ with $\lambda \leq 1/6$ small enough such that $F < \omega_k/k^{k/2}$. Let $T \in \mathbf{I}_k(X)$ be a cycle and define for $y \in X$

$$r_0(y) := \max\left\{r \geq 0 : \|T\|(B(y,r)) \geq Fr^k\right\}.$$

By [K, Theorem 9] we have

$$\lim_{r\searrow 0}\frac{\mathcal{H}^k(S_T \cap B(y,r))}{\omega_k r^k} = 1$$

for $\mathcal{H}^k$-almost all $y \in S_T$, the set S_T being defined as in (7). Together with Theorem 2.6 this implies that the set Y of points $y \in S_T$ satisfying $r_0(y) > 0$ has full $\|T\|$-measure. By Lemma 3.2 there exist points $y_1, \dots, y_N \in Y$ with $r_0(y_i) > 0$, and such that the balls $B(y_i, 2r_0(y_i))$ are pairwise disjoint and satisfy

$$\sum_{i=1}^{N}\|T\|\big(B(y_i, r_0(y_i))\big) \geq \alpha\|T\|(Y) = \alpha\mathbf{M}(T) \tag{8}$$

for a constant $\alpha > 0$ depending only on k. Fix $i \in \{1, \dots, N\}$ and set $r_0 := r_0(y_i)$ and $\beta(r) := \|T\|(B(y_i, r))$. It is clear that β is non-decreasing, that $\beta(r_0) = Fr_0^k$, and that $\beta(r) < Fr^k$ for all $r > r_0$. Denote furthermore by ϱ the function $\varrho(x) := d(y_i, x)$. By Theorem 2.7 the slice $\langle T, \varrho, r\rangle = \partial(T \llcorner B(y_i, r))$ exists for almost all r, is an element of $\mathbf{I}_{k-1}(X)$, and satisfies

$$\mathbf{M}\big(\langle T, \varrho, r\rangle\big) \leq \beta'(r) \quad \text{for a.e. } r\,. \tag{9}$$

We now consider one dimensional and higher dimensional cycles separately: If $k = 1$ it follows from the fact that $F = 1$ and from the definition of β that there exists a measurable set $\Omega \subset [r_0, 2r_0)$ of positive measure and such that $\beta'(r) < 1$ for all $r \in \Omega$. Since, for $r \in \Omega$, the slice $\langle T, \varrho, r\rangle$ is a 0-dimensional integral current, $\mathbf{M}(\langle T, \varrho, r\rangle)$ is an integer number and hence, by (9), the integral current $T_i := T \llcorner B(y_i, r)$ has zero boundary. Applying this to each $i \in \{1, \dots, N\}$ one easily obtains a decomposition $T = T_1 + \cdots + T_N + R$ satisfying all the properties stated in the proposition (with $\lambda = 0$, $\delta \geq \alpha$, and $E < 4$).

If $k \geq 2$ then Lemma 3.3 and the definitions of F and r_0 imply the existence of $\Omega \subset [r_0, 4r_0/3]$ of positive measure such that

$$C\big[\beta'(r)\big]^{k/(k-1)} < \lambda\beta(r) \quad \text{for all } r \in \Omega\,. \tag{10}$$

By Theorem 2.7 we can assume without loss of generality that the slice $\langle T, \varrho, r\rangle$ exists for every $r \in \Omega$ and is an element of $\mathbf{I}_{k-1}(X)$. Choose an $r \in \Omega$ arbitrarily and a filling $S \in \mathbf{I}_k(X)$ of $\langle T, \varrho, r\rangle$ as in Lemma 3.4.

Together with (9) and (10) this implies that

$$\mathbf{M}(S) \leq \lambda\beta(r)\,, \tag{11}$$

and, furthermore, since $\lambda \leq 1/6$, that the support of S lies in the ball with center y_i and with a radius $\bar{r}$ satisfying

$$\bar{r} \leq \frac{4}{3}r_0 + 3Ck\big[\mathbf{M}(\langle T, \varrho, r\rangle)\big]^{\frac{1}{k-1}} \leq \frac{4}{3}\left(1 + \frac{3Ck(\lambda F)^{1/k}}{C^{1/k}}\right)r_0 \leq 2r_0\,.$$

Clearly, $T_i := T \llcorner B(y_i, r) - S$ defines an integral cycle which satisfies

$$(1-\lambda)\beta(r) \leq \mathbf{M}(T_i) \leq (1+\lambda)\beta(r)\,.$$

Since T_i has support in $B(y_i, 2r_0(y_i))$ it follows that

$$\operatorname{diam}(\operatorname{spt} T_i) \leq 4r_0(y_i) = \frac{4}{F^{1/k}}\big[\beta(r_0)\big]^{1/k} \leq \frac{4}{[F(1-\lambda)]^{1/k}}\mathbf{M}(T_i)^{1/k}$$

and hence T_i fulfills condition (i) with $E := 4/[F(1-\lambda)]^{1/k}$.

Since our construction of T_i leaves $T \llcorner B^c(y_i, 2r_0(y_i))$ unaffected (by the fact that the balls $B(y_j, 2r_0(y_j))$ are pairwise disjoint) we can apply the above construction to every $i \in \{1, \dots, N\}$ to obtain round cycles $T_1, \dots, T_N$. Setting $R := T - \sum_{i=1}^N T_i$ this yields a decomposition $T = T_1 + \cdots + T_N + R$ satisfying the claimed properties. Indeed, we have

$$\sum_{i=1}^N \mathbf{M}(T_i) \leq (1+\lambda)\sum_{i=1}^N \|T\|(B_i) \leq (1+\lambda)\mathbf{M}(T)$$

where B_i is the ball chosen individually for every i as above. The estimate for $\mathbf{M}(R)$ is also obvious since, by (8) and (11), we have

$$\mathbf{M}(R) \leq \|T\|\Big(X \backslash \bigcup B_i\Big) + \lambda \sum \|T\|(B_i) \leq \big(1 - \alpha(1-\lambda)\big)\mathbf{M}(T)\,.$$

This completes the proof of the proposition with $\delta := \alpha(1-\lambda)$. □

The isoperimetric inequality now easily follows from Proposition 3.1.

Proof of Theorem 1.2. Let $T \in \mathbf{I}_k(X)$ be a cycle. Successive application of Proposition 3.1 yields (possibly finite) sequences of cycles (T_i), $(R_n) \subset \mathbf{I}_k(X)$ and an increasing sequence $(N_n) \subset \mathbb{N}$ with the following properties:

- $T = \sum_{i=1}^{N_n} T_i + R_n$;
- $\operatorname{diam}(\operatorname{spt} T_i) \leq E\mathbf{M}(T_i)^{1/k}$;
- $\mathbf{M}(R_n) \leq (1-\delta)^n \mathbf{M}(T)$;
- $\sum_{i=1}^\infty \mathbf{M}(T_i) \leq \big[(1+\lambda)\sum_{i=0}^\infty (1-\delta)^i\big]\,\mathbf{M}(T) = \frac{1+\lambda}{\delta}\mathbf{M}(T)$.

The isoperimetric filling of T is then constructed as follows. We first fill each T_i with an $S_i \in \mathbf{I}_{k+1}(X)$ from the cone inequality, i.e. one with $\partial S_i = T_i$ and such that

$$\mathbf{M}(S_i) \leq C_k \operatorname{diam}(\operatorname{spt} T_i)\mathbf{M}(T_i) \leq C_k E \mathbf{M}(T_i)^{(k+1)/k}. \tag{12}$$

The finiteness of $\sum_{i=1}^{\infty} \mathbf{M}(T_i)$ implies that the sequence $S^n := \sum_{i=1}^{N_n} S_i$ is a Cauchy-sequence with respect to the mass norm because

$$\mathbf{M}(S^{n+q} - S^n) \leq C_k E \sum_{i=N_n+1}^{\infty} \mathbf{M}(T_i)^{\frac{k+1}{k}} \leq C_k E \left[\sum_{i=N_n+1}^{\infty} \mathbf{M}(T_i) \right]^{\frac{k+1}{k}} \overset{n\to\infty}{\longrightarrow} 0 \,.$$

Since $\mathcal{I}_{k+1}(X)$ is a Banach space the sequence $S^n \in \mathbf{I}_{k+1}(X) \subset \mathcal{I}_{k+1}(X)$ converges to a limit current $S \in \mathcal{I}_{k+1}(X)$. As $T - \partial S^n = R_n$ converges to 0 it follows that $\partial S = T$ and, in particular, that $S \in \mathbf{I}_{k+1}(X)$. Finally, S is an isoperimetric filling of T. Indeed, we have

$$\mathbf{M}(S) \leq \sum \mathbf{M}(S_i) \leq C_k E \sum \mathbf{M}(T_i)^{\frac{k+1}{k}} \leq C_k E \left(\frac{1+\lambda}{\delta} \right)^{\frac{k+1}{k}} \mathbf{M}(T)^{\frac{k+1}{k}},$$

which completes the proof. □

We note that if T has bounded support then there exists an $S \in \mathbf{I}_{k+1}(X)$ as in the theorem and which, in addition, has bounded support. This follows directly from the remark after Lemma 3.4. Furthermore, if T has compact support then it is easy to prove, using Lemma 3.4, that there exists such an S with compact support. For this see also the second part of the proof of Theorem 1.6.

4 The Plateau Problem

Here we will solve the Plateau problem in generalized form and from this we will then derive Theorem 1.5 and Theorem 1.6.

DEFINITION 4.1. *A non-principal ultra-filter on* $\mathbb{N}$ *is a finitely additive probability measure* ω *on* $\mathbb{N}$ (*together with the* σ*-algebra of all subsets*) *such that* ω *takes values in* $\{0,1\}$ *only and* $\omega(A) = 0$ *whenever* $A \subset \mathbb{N}$ *is finite.*

Using Zorn's lemma it is not too hard to establish the existence of non-principal ultra-filters on $\mathbb{N}$. It is also easy to prove the following fact. If (Y, τ) is a compact topological Hausdorff space then for every sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ there exists a unique point $y \in Y$ such that

$$\omega\big(\{n \in \mathbb{N} : y_n \in U\}\big) = 1$$

for every $U \in \tau$ containing y. We will denote the point y by $\lim_\omega y_n$.

Now let (X, d) be a metric space and fix a non-principal ultra-filter ω on $\mathbb{N}$. We call a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ bounded if $\operatorname{diam}\{x_n\} < \infty$. An equivalence relation on the set of bounded sequences is given by

$$(x_n) \sim (x'_n) \quad \text{if and only if} \quad \lim_\omega d(x_n, x'_n) = 0\,.$$

Definition 4.2. *The ultra-completion $(X)_\omega$ of X is the set of equivalence classes of bounded sequences $(x_n)_{n\in\mathbb{N}}$ together with the metric given by*

$$d_\omega\big((x_n), (x'_n)\big) := \lim_\omega d(x_n, x'_n)$$

where $(x_n), (x'_n)$ are bounded sequences in X.

The space X can be isometrically embedded into $(X)_\omega$ by the map $\iota : X \to (X)_\omega$ assigning to x the constant sequence $(x)_{n\in\mathbb{N}}$. We are now in a position to state the main theorem of this section.

Theorem 4.3. *Let (X, d) be a complete metric space, $k \in \mathbb{N}$, and suppose X has an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(X)$. Let furthermore ω be a non-principal ultra-filter on $\mathbb{N}$. Then there exists for every $T \in \mathbf{I}_k(X)$ with $\partial T = 0$ and compact support an $S \in \mathbf{I}_{k+1}((X)_\omega)$ such that $\partial S = \iota_\# T$ and*

$$\mathbf{M}(S) \le \inf\left\{\mathbf{M}(S') : S' \in \mathbf{I}_{k+1}(X),\, \partial S' = T\right\}. \tag{13}$$

Remark 4.4. *It should be noted that S is a $(k+1)$-chain in the ultra-completion of X whereas the infimum in* (13) *is taken over all $(k+1)$-chains S' in X. In particular, this theorem does not claim the existence of a solution to the Plateau problem in the ultra-completion of X.*

The ideas of proof are similar to that of the weak*-compactness theorem for duals of separable Banach spaces in [AmK, Theorem 6.6]. The idea of using the ultra-completion of a metric space to attack the Plateau problem in the case of a Hadamard space is due to Urs Lang.

Proof. Let $T \in \mathbf{I}_k(X)$ be a cycle with compact support. We then claim that there exists a sequence $S_i \in \mathbf{I}_{k+1}(X)$ with the properties listed below:

(i) $\partial S_i = T$ for every $i \in \mathbb{N}$;
(ii) $\mathbf{M}(S_i) \longrightarrow \inf\{\mathbf{M}(S') : S' \in \mathbf{I}_{k+1}, \partial S' = T\}$ with $i \to \infty$;
(iii) The sequence $Z_i := \operatorname{spt} S_i$ is equi-bounded and equi-compact.

By Lemma 3.4 we find a sequence $(S_i) \subset \mathbf{I}_{k+1}(X)$ satisfying properties (i) and (ii) and such that for $x \in \operatorname{spt} S_i \backslash \operatorname{spt} T$ we have

$$\|S_i\|\big(B(x, r)\big) \ge \frac{r^{k+1}}{(3C)^k (k+1)^{k+1}} \quad \text{whenever } 0 \le r \le \operatorname{dist}(x, \operatorname{spt} T)\,. \tag{14}$$

We prove that the sequence of sets Z_i is equi-bounded and equi-compact. In order to do so fix $\varrho > 0$ and cover $\operatorname{spt} T$ by finitely many balls $B(y_m, \varrho)$, $m = 1, \dots, n$. We then choose points $x_1, x_2, \dots$ in $Z_i \backslash \cup B(y_m, 2\varrho)$ inductively such that the balls $B(x_m, \varrho/2)$ are pairwise disjoint. By (14) there exist only finitely many such points, say $x_1, \dots, x_N$. (Note that N can be chosen independent of i.) It follows that

$$Z_i \subset \bigcup_{m=1}^{N} B(x_m, 2\varrho) \cup \bigcup_{m=1}^{n} B(y_m, 2\varrho) =: U\,.$$

This shows the equi-compactness of the Z_i. To show equi-boundedness it is enough to note that, by minimality of S_i, there exists no connected component of U with empty intersection with $\operatorname{spt} T$. This establishes the existence of a sequence with the properties listed above.

By Gromov's compactness theorem there now exists a compact metric space (Z, d_Z) and (after extracting a subsequence) isometric embeddings $\varphi_i : Z_i \to Z$ such that the subsets $\varphi(Z_i)$ form a Cauchy sequence with respect to the Hausdorff metric on Z. Since Z is compact we may assume by Arzelà–Ascoli theorem that (after extracting a further subsequence) the $\varphi_i|_{\operatorname{spt} T}$ converge uniformly to an isometric embedding $\tilde{\varphi} : \operatorname{spt} T \to Z$. By the closure and compactness theorems for currents ([AmK, Theorems 5.2 and 8.5]) we may furthermore assume that $\varphi_{i\#} S_i$ converges weakly to a current $\hat{S} \in \mathbf{I}_{k+1}(Z)$. It follows easily that

$$\operatorname{spt} \hat{S} \subset \bigcap_{m=1}^{\infty} \overline{\bigcup_{i \geq m} \operatorname{spt}(\varphi_{i\#} S_i)} \subset \lim\nolimits_H \varphi_i(Z_i)$$

where $\lim_H$ denotes the Hausdorff limit for sequences of compact sets in Z. We define a map $\psi : Y := \lim_H \varphi_i(Z_i) \to (X)_\omega$ as follows: For $y \in Y$ there exists a sequence $z_i \in Z_i$ such that $\varphi(z_i) \to y$. We set $\psi(y) := (z_i)_{i \in \mathbb{N}}$. As is easily seen, the map ψ is well defined and an isometric embedding. Furthermore, we have for $S := \psi_\# \hat{S}$ that

$$\mathbf{M}(S) \leq \liminf \mathbf{M}(\varphi_{i\#} S_i) = \inf \left\{ \mathbf{M}(S') : S' \in \mathbf{I}_{k+1}(X),\, \partial S' = T \right\}.$$

It remains to show that $\partial S = \iota_\# T$. Clearly,

$$\partial \varphi_{i\#} S_i = \varphi_{i\#} T \to \tilde{\varphi}_\# T$$

and hence $\partial \hat{S} = \tilde{\varphi}_\# T$. Since, furthermore, $\psi \circ \tilde{\varphi} = \iota|_{\operatorname{spt} T}$ we obtain

$$\partial S = \partial \psi_\# \hat{S} = (\psi \circ \tilde{\varphi})_\# T = \iota_\# T\,,$$

concluding the proof. $\square$

From the above theorem we now derive Theorems 1.5 and 1.6. Of course, it suffices to prove the existence of a 1-Lipschitz retraction from the ultra-completion to the space itself. We first deal with the case of Hadamard spaces.

Proof of Theorem 1.6. If X is a Hadamard space then it follows easily that the ultra-completion $(X)_\omega$ is again a Hadamard space and that $\iota(X)$ is a closed convex subspace of $(X)_\omega$. Hence, by Proposition II.2.4 of [BH] there is a 1-Lipschitz retraction $\varphi : (X)_\omega \to \iota(X)$. If T and S are as in the above theorem then clearly $(\iota^{-1} \circ \varphi)_\#(S)$ is a minimal filling of T.
To prove the second statement of the theorem let $S \in \mathbf{I}_{k+1}(X)$ satisfy $\partial S = T$ and (3). It suffices to show that

$$\|S\|\big(B(x,r)\big) \geq \frac{r^{k+1}}{C^k(k+1)^{k+1}} \tag{15}$$

for every $x \in \operatorname{spt} S$ and $0 \leq r \leq \operatorname{dist}(x, \operatorname{spt} T)$, see the proof of Theorem 4.3. For this, fix $x \in \operatorname{spt} S$ and define $\beta(r) := \|S\|(B(x,r))$. Using the isoperimetric inequality and the slicing theorem it follows that

$$\beta(r) \leq C\big[\beta'(r)\big]^{k+1/k} \quad \text{for a.e. } r \in \big(0, \operatorname{dist}(x, \operatorname{spt} T)\big)\,,$$

from which (15) follows by applying Lemma 3.3. □

Proof of Theorem 1.5. As above, we show that there is a 1-Lipschitz retraction from $(E)_\omega$ onto E. We define a retraction $\varphi : (E)_\omega \to E$ by $\varphi((x_n)_n) := \lim_\omega x_n$. Note that $\lim_\omega x_n$ exists uniquely by the weak*-compactness of balls in E. We must check that φ is 1-Lipschitz. For this, let (x_n) and (y_n) be bounded sequences in E and fix $\varepsilon > 0$. We set $x := \lim_\omega x_n$ and $y := \lim_\omega y_n$ and choose an element z in the predual F of E with $\|z\| = 1$ and such that

$$\big|x(z) - y(z)\big| \geq \|x - y\| - \tfrac{\varepsilon}{4}\,.$$

By the definition of $\lim_\omega$ and the definition of the weak*-topology on E we obtain

$$\omega(A_1 \cap A_2 \cap A_3) = 1\,,$$

where $A_1 := \{m : |x_m(z) - x(z)| \leq \varepsilon/4\}$, $A_2 := \{m : |y_m(z) - y(z)| \leq \varepsilon/4\}$, and $A_3 := \{m : |\|x_m - y_m\| - d_\omega((x_n),(y_n))| \leq \varepsilon/4\}$. For an $m \in A_1 \cap A_2 \cap A_3$ we thus conclude

$$\|x - y\| \leq \big|x_m(z) - y_m(z)\big| + \tfrac{3\varepsilon}{4} \leq \|x_m - y_m\| + \tfrac{3\varepsilon}{4} \leq d_\omega\big((x_n),(y_n)\big) + \varepsilon\,.$$

Since $\varepsilon > 0$ was chosen arbitrarily this shows that φ is indeed 1-Lipschitz.

Finally, the second statement of the theorem follows exactly as in the proof of Theorem 1.6. □

REMARK 4.5. *For a general Banach space E there need not exist a 1-Lipschitz retraction from $(E)_\omega$ onto E. Indeed, an example is given by the space c_0 of real sequences tending to zero together with the supremum norm. (However, given a cycle in c_0 with compact support K one easily constructs a compact subset in c_0 containing K and which is a 1-Lipschitz retract of ℓ^∞. The same methods as above then show that the Plateau problem has a solution for compactly supported cycles in c_0.)*

References

[A] F.J. ALMGREN JR., Optimal isoperimetric inequalities, Indiana Univ. Math. J. 35 (1986), 451–547.

[AmK] L. AMBROSIO, B. KIRCHHEIM, Currents in metric spaces, Acta Math. 185:1 (2000), 1–80.

[BH] M.R. BRIDSON, A. HAEFLIGER, Metric Spaces of Non-Positive Curvature, Grundlehren der mathematischen Wissenschaften 319, Springer, 1999.

[C] C. CROKE, A sharp four dimensional isoperimetric inequality, Comment. Math. Helv. 59 (1984), 187–192.

[E] I. EKELAND, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.

[FF] H. FEDERER, W.H. FLEMING, Normal and integral currents, Ann. Math. 72 (1960), 458–520.

[G] M. GROMOV, Filling Riemannian manifolds, J. Diff. Geom. 18 (1983), 1–147.

[K] B. KIRCHHEIM, Rectifiable metric spaces: local structure and regularity of the Hausdorff measure, Proc. Am. Math. Soc. 121:1 (1994), 113–123.

[Kl1] B. KLEINER, An isoperimetric comparison theorem, Invent. Math. 108 (1992), 37–47.

[Kl2] B. KLEINER, The local structure of length spaces with curvature bounded above, Math. Z. 231:2 (1999), 409–456.

STEFAN WENGER, Department of Mathematics, ETH Zentrum, CH – 8092 Zürich, Switzerland
wenger@math.ethz.ch

Received: August 2003
Revision: June 2004
Accepted: August 2004